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Hamiltonian motions of plane curves and formation of singularities and bubbles

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Abstract

A class of Hamiltonian deformations of plane curves is defined and studied. Hamiltonian deformations of conics and cubics are considered as illustrative examples. These deformations are described by systems of hydrodynamical-type equations. It is shown that solutions of these systems describe processes of formation of singularities (cusps, nodes), bubbles and change of the genus of a curve.

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1. Introduction

Dynamics of curves and interfaces is a key ingredient in various important phenomena in physics and theories in mathematics (see, e.g. [3, 6, 26, 33]). The special class of deformations of curves described by integrable equations has attracted particular interest. It has been studied in a number of papers (see [4, 5, 7, 8, 10, 14, 17–25, 27–29, 31, 32, 39, 40] and references therein).

The approaches used in these papers differ, basically, in the way the evolution of a curve is fixed. In papers [7, 10, 14, 25, 27, 32], the motion of the curve is defined by the requirement that the time derivative of the position vector of a curve be a linear superposition of tangent and (bi)normal vectors with a certain specification of coefficients in this superposition. Deformations considered in papers [4, 8, 22, 23, 39] are defined, basically, by the corresponding Lax pair. Within the study of the Laplacian growth problem [5, 24, 28, 29, 31, 39, 40], the time evolution of the interface is characterized by the dynamics of the Schwartz function of a curve prescribed by the Darcy law. Semiclassical deformations of algebraic curves analyzed in [17–19] are fixed by the existence of a specific generating

function constructed via a Lenard scheme. Finally, the coisotropic deformations introduced in [21] are defined by the requirement that the ideal of the deformed curve be a Poisson ideal.

In this paper, a novel class of deformations of plane curves is considered. For such deformations, the function $f(p_1, p_2)$, which defines a curve $f(p_1, p_2) = 0$ in the plane (p_1, p_2) , obeys an inhomogeneous Liouville equation with p_1, p_2 and deformation parameters x_1, x_2, t playing the role of independent variables. These deformations are referred to as the Hamiltonian deformations (motions) of a curve since the coefficients in the linear equation are completely fixed by the single function $H(p_1, p_2; x_1, x_2, t)$ playing the role of a Hamiltonian for time t dynamics. Equivalently, Hamiltonian deformations can be defined as those for which an ideal \mathcal{I} of a deformed curve is invariant under the action of the vector field $\partial_t + \nabla_H$. For algebraic plane curves, Hamiltonian deformations represent a particular class of coisotropic deformations introduced in [21].

Here, we concentrate on the study of Hamiltonian dynamics of plane quadrics (conics) and cubics. This dynamics is described by integrable systems of hydrodynamical type for the coefficients of polynomials defining algebraic curves with the dKdV, dDS, dVN, (2+1)-dimensional one-layer Benney system and other equations among them. It is shown that particular solutions of these systems describe formation of singularities (cusps, nodes) and bubbles on the real plane.

This paper is organized as follows. A general definition and interpretations are given in section 2. Hamiltonian deformations of plane quadrics are studied in section 3. As particular examples, one has deformations of a circle described by the dispersionless Veselov–Novikov (dVN) equation and deformations of an ellipse given by a novel system of equations. Section 4 is devoted to Hamiltonian deformations of plane cubics. An analysis of formation of singularities for cubics and genus transition is given. In section 5, a connection between singular cubics and Burgers–Hopf equation is considered. In section 6, we discuss briefly deformations of a quintic. Possible extensions of the approach presented in this paper are noted in section 7.

2. Definition and interpretation

Let the plane curve Γ be defined by the equation

$$f(p_1, p_2) = 0, \tag{1}$$

where f is a function of local affine coordinates p_1, p_2 on the complex plane \mathbb{C}^2 . To introduce deformations of the curve (1), we assume that f depends also on the deformation parameters x_1, x_2, t .

Definition 1. *If the function $f(p_1, p_2; x_1, x_2, t)$ obeys the equations*

$$\partial_t f + \{f, H\} = \alpha(p_1, p_2; x_1, x_2, t)f \tag{2}$$

with a certain function $H(p_1, p_2; x_1, x_2, t)$ and a function α , where $\{f, H\}$ is the Poisson bracket

$$\{f, H\} = \sum_{i=1}^2 (\partial_{x_i} f \partial_{p_i} H - \partial_{p_i} f \partial_{x_i} H), \tag{3}$$

then it is said that the function f as a function of x_1, x_2 and t defines Hamiltonian deformations of the curve (1).

To justify this definition, we first observe that equation (2) is nothing but an inhomogeneous Liouville equation which is well known in classical mechanics. On Γ the rhs of equation (2) vanishes and, hence,

$$(\partial_t + \nabla_H)f|_{\Gamma} = 0, \tag{4}$$

where ∇_H is the Hamiltonian vector field in the space with coordinates p_1, p_2, x_1, x_2 . Thus, Hamiltonian deformations of the curve (1) generated by H can be equivalently defined as those for which the function f obeys the condition (4).

Following the standard interpretation of the Liouville equation in classical mechanics, one can view equations (2) and (4) as

$$\frac{df}{dt} = \alpha f \tag{5}$$

and

$$\left. \frac{df}{dt} \right|_{\Gamma} = 0, \tag{6}$$

where the total derivative $\frac{d}{dt}$ is calculated along the characteristics of equation (2) defined by Hamiltonian equations

$$\frac{dp_i}{dt} = -\partial_{x_i} H, \quad \frac{dx_i}{dt} = \partial_{p_i} H, \quad i = 1, 2. \tag{7}$$

For a given irreducible curve Γ ($f(p_1, p_2; x_1, x_2, t) = 0$) a function α is not defined uniquely. Indeed, for any function $\beta(p_1, p_2; x_1, x_2, t)$ the function $\hat{f} = \beta f$ defines the same curve (1). Then, equation (2) takes the form

$$\partial_t \hat{f} + \{\hat{f}, H\} = \hat{\alpha} \hat{f}, \tag{8}$$

where

$$\beta \hat{\alpha} = \alpha + \partial_t \beta + \{\beta, H\} \tag{9}$$

while the condition (4) remains unchanged. Thus, the freedom in the form of α given by (9) is just a gauge-type freedom. For given α , the choice of β as a solution of the equation

$$\beta \alpha = -\partial_t \beta - \{\beta, H\} \tag{10}$$

gives $\hat{\alpha} = 0$. So, there exists a gauge at which equation (2) is homogeneous one. The drawback of such a ‘gauge’ is that in many cases equation (1) has no simple, standard form in this gauge.

Gauge freedom observed above becomes very natural in the formulation based on the notion of ideal of a curve. Ideal $\mathcal{I} = \langle f \rangle$ of an irreducible curve (1) is defined as a set of all functions vanishing on the curve (1) (see e.g. [13]). For any function g without singularities on Γ the product gf belongs to \mathcal{I} . In these terms, Hamiltonian deformations of a curve can be characterized equivalently by the following.

Definition 2. *Hamiltonian deformations of a curve generated by a function H are those for which the ideal \mathcal{I} of the deformed curve is invariant under the action of the vector field $\partial_t + \nabla_H$, i.e.*

$$(\partial_t + \nabla_H)\mathcal{I} \subset \mathcal{I}. \tag{11}$$

The definition of the Hamiltonian deformation in this form clearly shows the irrelevance of the concrete form of function α in equation (2). Moreover, such formulation reveals also the nonuniqueness in the form of H for the same deformation. Indeed, for the family of Hamiltonians $\tilde{H} = H + G$ where G is an arbitrary element of \mathcal{I} (i.e. $g \cdot f$), one has equation (2) with the family of functions $\hat{\alpha}$ of form $\tilde{\alpha} = \alpha + \{f, g\}$. Hence, equations $(\partial_t + \nabla_{\tilde{H}})f|_{\Gamma} = 0$ and $(\partial_t + \nabla_H)f|_{\Gamma} = 0$ and the corresponding equations (11) coincide. This freedom ($H \rightarrow H + gf$ where $g(p_1, p_2; x_1, x_2, t)$ is an arbitrary function) in the choice of Hamiltonians is associated with different possible evaluations of H on the curve Γ . In practice, it corresponds to different

possible resolutions of the concrete equation (1) with respect to one of the variables (p_1 or p_2). This freedom serves to choose the simplest and most convenient form of Hamiltonian H .

For algebraic curves, i.e. for functions f polynomial in p_1, p_2 , Hamiltonian deformations of plane curves represent a special subclass of coisotropic deformations of algebraic curves in three-dimensional space considered in [21].

In three-dimensional space, the curve Γ is defined by the system of two equations

$$f_1(p_1, p_2, p_3, x_1, x_2, x_3) = 0, \quad f_2(p_1, p_2, p_3, x_1, x_2, x_3) = 0, \quad (12)$$

where x_1, x_2, x_3 are deformation parameters. Its coisotropic deformation is fixed by the condition [21]

$$\{f_1, f_2\}|_{\Gamma} = 0, \quad (13)$$

where $\{ , \}$ is the canonical Poisson bracket in \mathbb{R}^6 . It is easy to see that with the choices $f_1 = f(p_1, p_2; x_1, x_2, x_3)$ and $f_2 = p_3 + H(p_1, p_2; x_1, x_2, x_3)$, the condition (13) is reduced to (4) where $t = x_3$. In general, Hamiltonian deformations of the plane curve are parameterized by three independent variables x_1, x_2 and t . If one variable is cyclic (say x_2 , i.e. $\partial_{x_2} H = 0$ and, hence, $p_2 = \text{const}$) then deformations are described by $(1 + 1)$ -dimensional equations.

For Hamiltonian deformations of complex curves defined above, the deformation parameters and all other variables are complex numbers. Exactly in the same manner, one can define the Hamiltonian deformations of real curves on \mathbb{R}^2 .

3. Deformations of plane quadrics

In the rest of the paper, we will consider Hamiltonian deformations of complex algebraic plane curves. All equations and formulae derived in this and subsequent sections are valid in both complex and real cases. As far as the deformations presented in the figures are concerned, then, of course, the deformation parameters are real. So, the figures visualize Hamiltonian deformations of real plane curves or, equivalently, of the real sections of complex curves.

We begin with quadrics defined by the equation

$$f = ap_1^2 + bp_2^2 + cp_1p_2 + dp_1 + ep_2 + h = 0, \quad (14)$$

where a, b, \dots, h are the functions of the deformation parameters x_1, x_2, t . Choosing

$$H = \alpha p_1^2 + \beta p_2^2 + \gamma p_1p_2 + \delta p_1 + \mu p_2 + \nu, \quad (15)$$

one obtains a system of six equations of hydrodynamical type. This system has several interesting reductions. One is associated with the constraint $a = b = e = 0, c = 1, \gamma = \mu = 0$, and α and β are constants. It is given by the system

$$\begin{aligned} d_t + \delta d_{x_1} - \beta(d^2)_{x_2} + 2\alpha h_{x_1} - \nu_{x_2} &= 0, \\ h_t + (h\delta)_{x_1} - 2\beta(dh)_{x_2} &= 0, \\ \delta_{x_2} - 2\alpha d_{x_1} &= 0, \\ \nu_{x_1} - 2\beta h_{x_2} &= 0. \end{aligned} \quad (16)$$

At $\delta = 0, \alpha = 0, \beta = \frac{1}{2}$ it is the $(2 + 1)$ -dimensional generalization of the one-layer Benney system proposed in [23, 41]

$$\begin{aligned} d_{x_1t} - \frac{1}{2}(d^2)_{x_1x_2} - h_{x_2x_2} &= 0, \\ h_t - (dh)_{x_2} &= 0. \end{aligned} \quad (17)$$

The curve Γ is the hyperbola $p_1p_2 + dp_1 + h = 0$ and the solutions of the system (17) describe deformations of the hyperbola. The simplest solution of the system is given by

$$d = t, \quad h = x_2 + \frac{t^2}{2}. \quad (18)$$

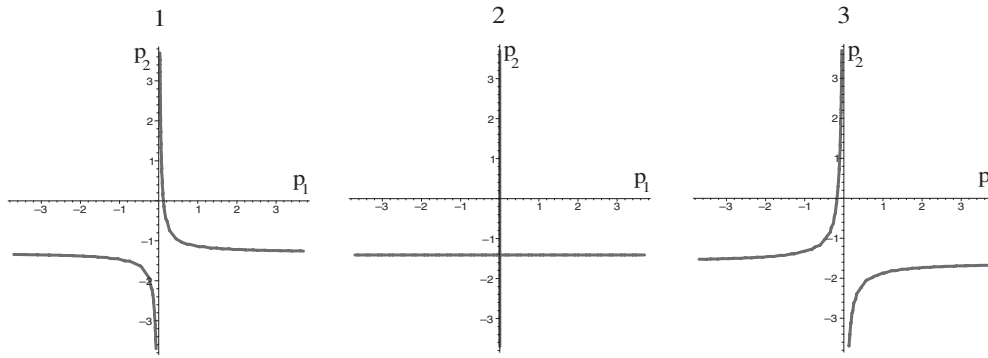


Figure 1. First plot: $x_2 = -1, t = 1.3$; second plot: $x_2 = -1, t = \sqrt{2} = 1.4142$; third plot: $x_2 = -1, t = 1.6$.

The corresponding evolution of the hyperbola is shown in figure 1.

At $\beta = -\alpha = \frac{1}{2}$, the system (16) becomes the dispersionless Davey–Stewardson (dDS) system considered in [18]. It describes a class of Hamiltonian motions of a hyperbola.

For cubic H given by

$$H = \alpha_1 p_1^3 + \alpha_2 p_2^3 + \alpha_3 p_1^2 p_2 + \alpha_4 p_2^2 p_1 + \alpha_5 p_1^2 + \alpha_6 p_2^2 + \alpha_7 p_1 p_2 + \alpha_8 p_1 + \alpha_9 p_2 + \alpha_{10} \tag{19}$$

the condition (4) gives rise to a quite long system of equations. This system admits several distinguished reductions.

Under the constraint $b = c = d = 0, a = e = 1, \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = \alpha_6 = \alpha_7 = \alpha_9 = 0, \alpha_1 = 1$ we obtain

$$\begin{aligned} h_t + \frac{3}{2} h_{x_1} h - \alpha_{10x_2} &= 0, \\ 3h_{x_2} + 4\alpha_{10x_1} &= 0, \end{aligned} \tag{20}$$

which is the well-known dispersionless Kadomtsev–Petviashvili (dKP) equation (see e.g. [16, 22, 41]). It describes Hamiltonian deformations of a parabola $p_1^2 + p_1 + h = 0$.

For the reduction $a = b = 1, c = d = e = 0, \alpha_2 = \alpha_4 = \alpha_5 = \alpha_6 = \alpha_7 = \alpha_{10} = 0, \alpha_1 = -\frac{1}{3}\alpha_3 = 1$ one has the system

$$\begin{aligned} h_t + (h\alpha_8)_{x_1} + (h\alpha_9)_{x_2} &= 0, \\ 3h_{x_1} - \alpha_{8x_1} + \alpha_{9x_2} &= 0, \\ 3h_{x_2} + \alpha_{8x_2} + \alpha_{9x_1} &= 0 \end{aligned} \tag{21}$$

that is the dVN equation [22]. It gives us Hamiltonian deformations of a circle $p_1^2 + p_2^2 + h = 0$.

Another interesting reduction describes deformations of the quadric (an ellipse in the real case)

$$\frac{p_1^2}{a^2} + \frac{p_2^2}{b^2} - 1 = 0, \tag{22}$$

i.e. the quadric (14) with $a \rightarrow \frac{1}{a^2}, b \rightarrow \frac{1}{b^2}, c = d = e = 0$ and $h = -1$. Considering particular Hamiltonian (19) with $\alpha_3 = \alpha_4 = \alpha_5 = \alpha_6 = \alpha_7 = \alpha_{10} = 0$, one has the following

new system of equations:

$$\begin{aligned}
 a_t + (\alpha_8 a + \alpha_1 a^3)_{x_1} + \alpha_9 a_{x_2} &= 0, \\
 b_t + (\alpha_9 b + \alpha_2 b^3)_{x_2} + \alpha_8 b_{x_1} &= 0, \\
 a^4 \alpha_{1x_2} - b^4 \alpha_{2x_1} &= 0, \\
 a^4 \alpha_{1x_2} + a^2 \alpha_{8x_2} + b^2 \alpha_{9x_1} &= 0, \\
 a(\alpha_2 b^3)_{x_2} + b(\alpha_1 a^3)_{x_1} - 3\alpha_1 a^3 b_{x_1} - 3\alpha_2 b^3 a_{x_2} &= 0.
 \end{aligned} \tag{23}$$

For the system (23), equation (2) is of form

$$f_t + \{f, H\} = (A p_1^2 + B p_2^2 + C p_1 p_2) f, \tag{24}$$

where

$$A = -2\alpha_{1x_1} - 6\frac{\alpha_1 a_{x_1}}{a}, \quad B = -2\alpha_{2x_2} - 6\frac{\alpha_2 b_{x_2}}{b}, \quad C = -2\frac{a^2}{b^2} \alpha_{1x_1}. \tag{25}$$

Solutions of the system (23) describe various regimes in Hamiltonian motions of the ellipse. In particular, for the initial data given by $a(x_1, x_2, t = 0) = b(x_1, x_2, t = 0)$ one has deformations of a circle into an ellipse.

For $\alpha_1 = \text{const}$ and $\alpha_2 = \text{const}$ the system (23) admits the reduction $a = b$. At $\alpha_1 = 4, \alpha_2 = 0$ the corresponding system coincides with the dVN equation (21) modulo the substitutions $h = -a^2$ and $\alpha_8 \rightarrow 3h + \alpha_8$. We note that the last change of α_8 is suggested by the transformation of type $H \rightarrow H + gf$ discussed in section 2 from the Hamiltonian H for the system (23) to the Hamiltonian for the dVN equation.

In the case of cyclic variable x_2 , the last three equations (23) give

$$\alpha_2 = \text{const}, \quad \alpha_9 = \text{const}, \quad \alpha_1 = \frac{b^3}{a^3}, \tag{26}$$

while the first two equations take the form ($x = x_1$)

$$\begin{aligned}
 a_t + (\alpha_8 a + b^3)_x &= 0, \\
 b_t + \alpha_8 b_x &= 0.
 \end{aligned} \tag{27}$$

The Hamiltonian becomes $H = \frac{b^3}{a^3} p_1^3 + \alpha_8 p_1$. Without loss of generality we choose $\alpha_2 = \alpha_9 = 0$. Introducing new variables $u = \frac{a}{b}$ and $v = b^2$, one rewrites the (1+1)-dimensional system (27) as

$$\begin{aligned}
 u_t + (\alpha_8 u + \frac{3}{2}v)_x &= 0, \\
 v_t + \alpha_8 v_x &= 0.
 \end{aligned} \tag{28}$$

Finally, using the eccentricity $\epsilon = \sqrt{1 - u^2}$ of an ellipse, one obtains the system

$$\begin{aligned}
 \epsilon_t - \frac{\sqrt{1 - \epsilon^2}}{\epsilon} \left(\alpha_8 \sqrt{1 - \epsilon^2} + \frac{3}{2}v \right)_x &= 0, \\
 v_t + \alpha_8 v_x &= 0,
 \end{aligned} \tag{29}$$

which describes Hamiltonian deformations of an ellipse in a pure geometrical term.

The system (28) represents a particular example of the two component (1+1)-dimensional systems of hydrodynamical type (with arbitrary function $\alpha_8(u, v)$). It is well known that such systems are linearizable by a hodograph transformation $x = x(u, v), t = t(u, v)$ (see e.g. [8]). In our case, the corresponding linear system is

$$\begin{aligned}
 x_u - \alpha_8 t_u &= 0, \\
 x_v + \left(\frac{3}{2} + u\alpha_{8v} \right) t_u - (\alpha_8 + u\alpha_{8u}) t_v &= 0.
 \end{aligned} \tag{30}$$

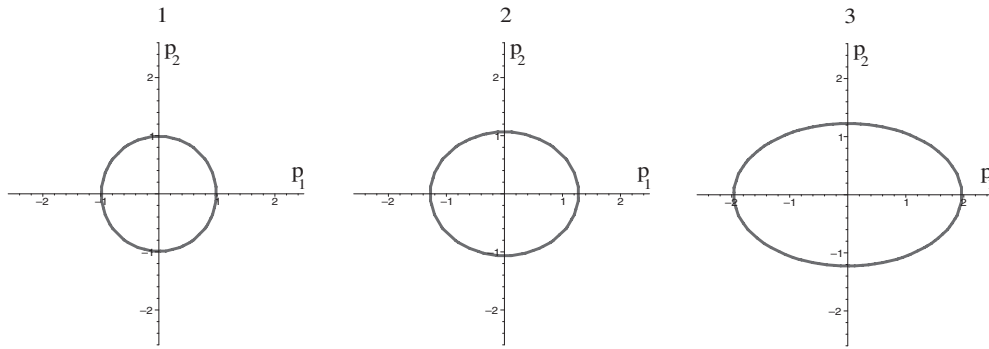


Figure 2. First plot: $x_1 = -1, t = 1$; second plot: $x_1 = -1, t = 1.2$; third plot: $x_1 = -1, t = 1.6$.

This fact allows us to construct explicitly a wide class of solutions for the system (28).

At $\alpha_8 = u$ the system (28) is the dispersionless limit of the Hirota–Satsuma system [15]. It has a simple polynomial solution

$$\begin{aligned} u &= t, \\ v &= -\frac{2}{3}x + \frac{1}{3}t^2, \end{aligned} \tag{31}$$

which provides us with deformation of a circle ($t = 1$) into an ellipse shown in figure 2.

4. Deformations of plane cubic curve

Hamiltonian deformations of plane cubics exhibit much richer and interesting structure.

The general form of the plane cubic is (see e.g. [13])

$$f = p_3^2 - p_2^3 - u_4 p_3 p_2 - u_3 p_2^2 - u_2 p_3 - u_1 p_2 - u_0. \tag{32}$$

(1+1)- and (2+1)-dimensional coisotropic deformations of plane cubics have been considered in the paper [21], where the corresponding hydrodynamical-type system has been derived. Here, we will concentrate on the analysis of the formation of singularities and bubbles for the real section of cubic curves described by particular solutions of these systems. To this end, we restrict ourselves to the (1+1)-dimensional deformations which correspond to the case of cyclic variable x_2 and the reduction $u_4 = u_2 = 0$. Thus, denoting $p_2 = z$ and $p_3 = p$, one has the cubic curve

$$p^2 - (z^3 + u_3 z^2 + u_1 z + u_0) = 0. \tag{33}$$

The particular choice of H as

$$H = \left(\frac{u_3}{2} - z\right) p \tag{34}$$

gives rise to the following system ($x = x_1$):

$$\begin{aligned} u_{3t} &= u_{1x} - \frac{3}{2}u_3 u_{3x}, \\ u_{1t} &= u_{0x} - u_1 u_{3x} - \frac{1}{2}u_3 u_{1x}, \\ u_{0t} &= -u_0 u_{3x} - \frac{1}{2}u_3 u_{0x}, \end{aligned} \tag{35}$$

which is the well-known three-component dispersionless Korteweg–de Vries (dKdV) system (see [9, 17]). The system (35) describes the Hamiltonian (coisotropic) deformations of the

curve (33). Recall that the moduli of the elliptic curve (33) are given by [37]

$$\begin{aligned} g_2 &= u_1 - \frac{1}{3}u_3^2, \\ g_3 &= u_0 + \frac{2}{27}u_3^3 - \frac{1}{3}u_3u_1 \end{aligned} \tag{36}$$

while the discriminant Δ is

$$\Delta = -16(4g_2^3 + 27g_3^2) = 16u_3^2u_1^2 - 64u_1^3 - 432u_0^2 - 64u_3^3u_0 + 288u_0u_3u_1. \tag{37}$$

The elliptic curve has singular points and, hence, becomes rational (genus zero) at the points where $\Delta = 0$.

We also note that in this case equation (2) is

$$\partial_t f + \{f, H\} = -u_{3x}f. \tag{38}$$

We will analyze polynomial solutions of the system (35). The simplest one, linear in x and t , is

$$\begin{aligned} u_3 &= 2a, \\ u_1 &= a^2 + 2b + 2d t, \\ u_0 &= 2(ab + c) + 2d x - 2ad t, \end{aligned} \tag{39}$$

where a, b, c, d are arbitrary constants. With the choice $a = b = c = 0$ and $d = \frac{1}{2}$ one has

$$u_3 = 0, \quad u_1 = t, \quad u_0 = x. \tag{40}$$

Equation (33) in this case is

$$p^2 - z^3 - tz - x = 0 \tag{41}$$

and the discriminant becomes

$$\Delta = -16(4t^3 + 27x^2). \tag{42}$$

So, for this solution deformation parameters t and x coincide with moduli g_2 and g_3 .

On the curve

$$4t^3 + 27x^2 = 0 \tag{43}$$

in the plane (x, t) (figure 3) which has the standard parameterization

$$x = 2s^3, \quad t = -3s^2, \tag{44}$$

the elliptic curve (33) is singular, i.e.

$$p^2 - (z - s)^2(z + 2s) = 0. \tag{45}$$

The curve (43) divides the plane (t, x) into two different domains (figure 3). In domain D, the discriminant $\Delta > 0$ and, hence, the real section of the elliptic curve is disconnected while in domain C ($\Delta < 0$) it is connected. The genus of the curve in both domains is equal to 1. On the curve (43) it vanishes.

Numerical analysis shows that the transition from the connected to the disconnected curve (and vice versa) may happen in three qualitatively different ways shown in figures 4–6 depending on the sign of x on the curve (43). Figure 4 represents the evolution of the real section of the curve (41) in the case when deformation parameters t, x change along the line $x = 0.2$. Transition from the connected to the disconnected (with bubble) curve goes through the formation of a double point (node) at $t = -0.64633$. At $x = -0.2$ (figure 5), the bubble grows up from the point which appears at $t = 3 \times 10^{-2/3}$. From the complex viewpoint, the above two regimes are the same. They are related to each other by the involution $x \rightarrow -x$

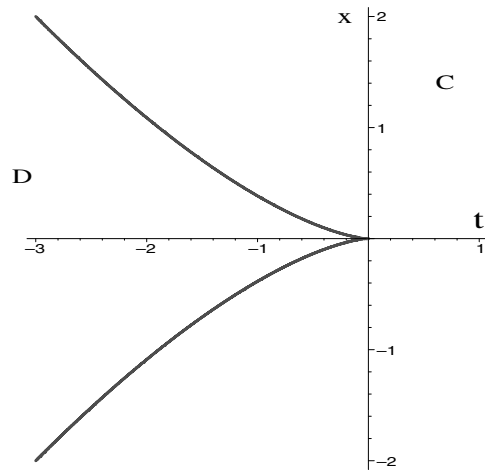


Figure 3. On the curve, the genus of the cubic (41) is 0, outside the curve it is 1. In the D region, the real section of the elliptic curve is disconnected, in the C region it is connected.

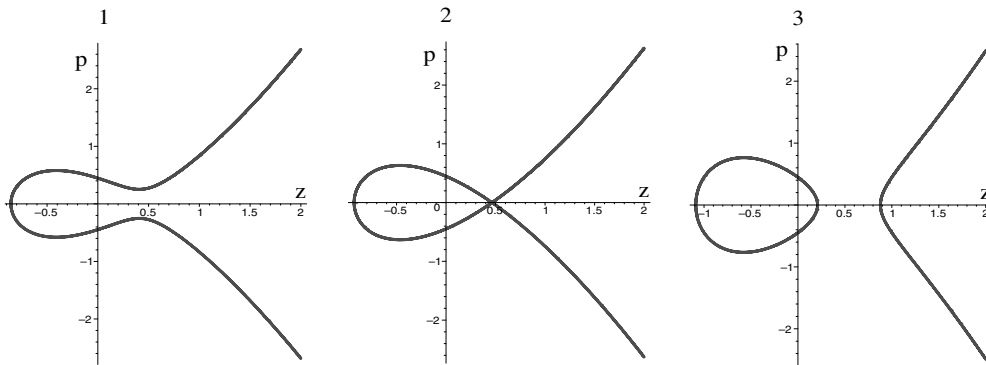


Figure 4. First plot: $x = 0.2, t = -0.5$; second plot: $x = 0.2, t = -3 \times 10^{-2/3} = -0.64633$; third plot: $x = 0.2, t = -1$.

of the curve (41). Transition through the line $x = 0$ at $t = 0$ is the limiting case of the above transitions. The formation (or annihilation) of a bubble goes through the formation of the cusp $p^2 - z^3 = 0$ (figure 6). In figures 4–6 and others the plots are ordered in a way to describe transition from the connected to the disconnected curve. The processes of annihilation and creation of a bubble are converted to each other by a simple change of sign of Hamiltonian.

Richer transition phenomena are observed for the solution (39) with $a = b = c = -d = 1$, i.e.

$$u_3 = 2, \quad u_1 = 3 - 2t, \quad u_0 = 4 + 2(t - x). \tag{46}$$

In this case, the curve is given by

$$p_3^2 - z^3 - 2z^2 - (3 - 2t)z - 2t - 4 + 2x = 0 \tag{47}$$

and the discriminant is

$$\Delta = -64(50 + 95t^2 - 70x + 100t - 90xt - 8t^3 + 27x^2). \tag{48}$$

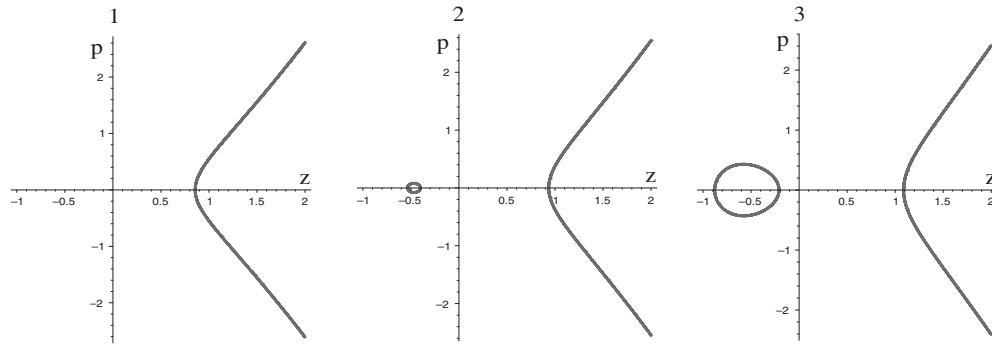


Figure 5. First plot: $x = -0.2, t = -0.5$; second plot: $x = -0.2, t = -0.66$; third plot: $x = -0.2, t = -1$. The bubble becomes a point in $z = -0.46416$ at $t = -3 \times 10^{-2/3}$.

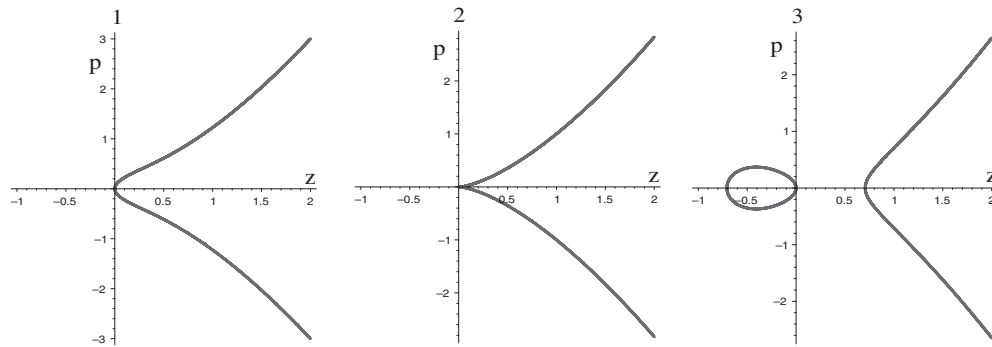


Figure 6. First plot: $x = 0, t = 0.5$; second plot: $x = 0, t = 0$; third plot: $x = 0, t = -0.5$.

The ‘phase diagram’ (with the curve $\Delta = 0$) is given in figure 7. Transitions across the curve $\Delta = 0$ at $x > x_M$ and $x < x_m$ are qualitatively the same as shown in figures 4 and 5. A new regime occurs for the transition points with $x_m < x < x_M$. In this case changing deformation parameters along the line $x = \text{const}$ one has the process of formation of the bubble then its absorption and again formation (figure 8). A new behavior appears in this evolution. Starting from a connected curve, a bubble is born and grows far from the boundary until it is connected with the wall through a node. After that the node is desingularized and the real section of the curve returns connected. Then, always through a node, a bubble is generated again by the boundary and never comes back.

Higher order polynomial solution of the system (35) is given by

$$\begin{aligned} u_3 &= 2A + 2Et, \\ u_1 &= -E^2 t^2 + 2Dt + 2Ex + A^2 + 2B, \\ u_0 &= -(AE^2 + ED) t^2 - (2AD + 2A^2E) t + 2AE x + 2Dx + 2C + 2AB, \end{aligned} \tag{49}$$

where A, B, C, D, E are constants.

For the particular choice $A = B = C = -D = -E = 1$ the elliptic curve becomes

$$p^2 - z^3 - (2 - 2t)z^2 - (-t^2 - 2t - 2x + 3)z - (-2t^2 + 4t - 4x + 4) = 0 \tag{50}$$

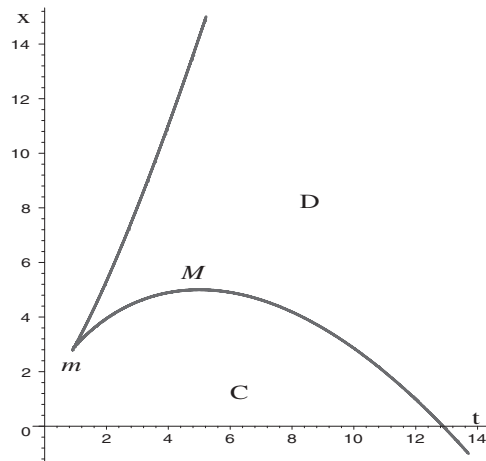


Figure 7. On the line, the genus of the curve (47) is 0, outside the line it is 1. In the D region, the real section of the elliptic curve is disconnected, in the C region it is connected. The point M is the local maximum of the curve and the point m is a local minimum (cusp).

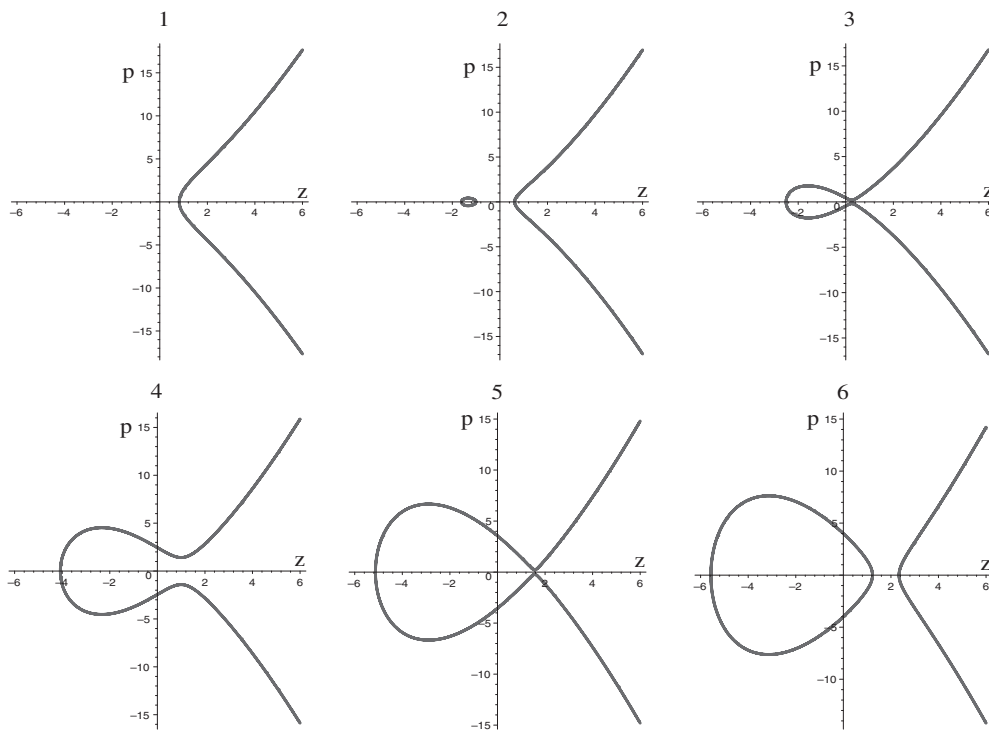


Figure 8. First plot: $x = 4, t = -1$; second plot: $x = 4, t = 1.5$; third plot: $x = 4, t = 2.1$; fourth plot: $x = 4, t = 5$; fifth plot: $x = 4, t = 8.3$; sixth plot: $x = 4, t = 10$.

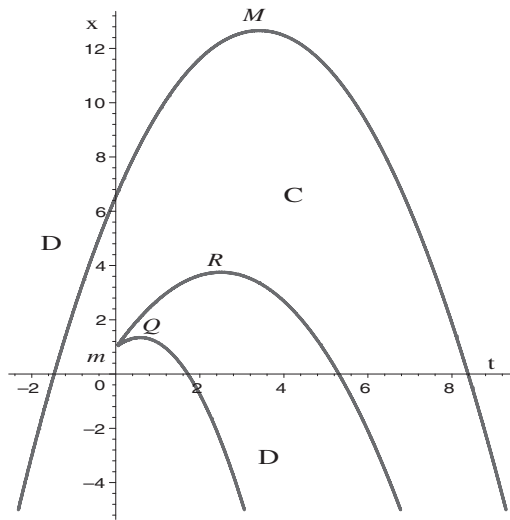


Figure 9. On the curve, the genus of the cubic (47) is 0, outside the curve it is 1. In the D regions, the real section of the elliptic curve is disconnected, in the C region it is connected. The point M is the global maximum of the curve, the points Q and R are local maxima and the point m is a local minimum (cusp).

and the discriminant is

$$\Delta = 16(-200 + 260t^4 - 272x^2 - 800t + 440x - 104t^5 + 64t^2x^2 + 8t^6 + 40t^4x - 320t^3x - 224x^2t + 32x^3 + 1040xt + 96t^2x - 640t^2 + 504t^3). \tag{51}$$

The ‘phase diagram’ containing the six-order curve $\Delta = 0$ is shown in figure 9.

For such deformation of the elliptic curve, one observes much richer structure of transition regimes. At $x > x_M$, the curve remains disconnected for all values of deformation parameters. At $x_M > x > x_R$, the transition involves the absorption and the recreation of a bubble. For $x_R > x > x_Q$, the transition involves a double absorption and creation of a bubble. At $x < x_m$, one has the replication of this regime. For $x_Q > x > x_m$, the transition process goes through a complicated oscillation depicted in figure 10 of the bubble interacting with the wall by means of many different nodal critical points. The above simple examples demonstrate the richness of possible transition processes between the connected and disconnected (with a bubble) real sections of the elliptic curve described by the system (35).

5. Singular elliptic curve and Burgers–Hopf equation

For all processes of formation and annihilation of bubbles described in the previous section, the elliptic curve passes through the ‘singular point’ x^*, t^* with $\Delta(x^*, t^*) = 0$ at which the curve becomes degenerated (rational). For example, for the simplest solution (40) the elliptic curve assumes the form (45) for the deformation parameters obeying equation (43). The cubic curve (45) is degenerated and possesses a double point for $s \neq 0$ and a cusp for $s = 0$ (see e.g. [37]).

More generally, an elliptic curve written in terms of moduli, i.e.

$$p^2 - (z^2 + g_2z + g_3) = 0, \tag{52}$$

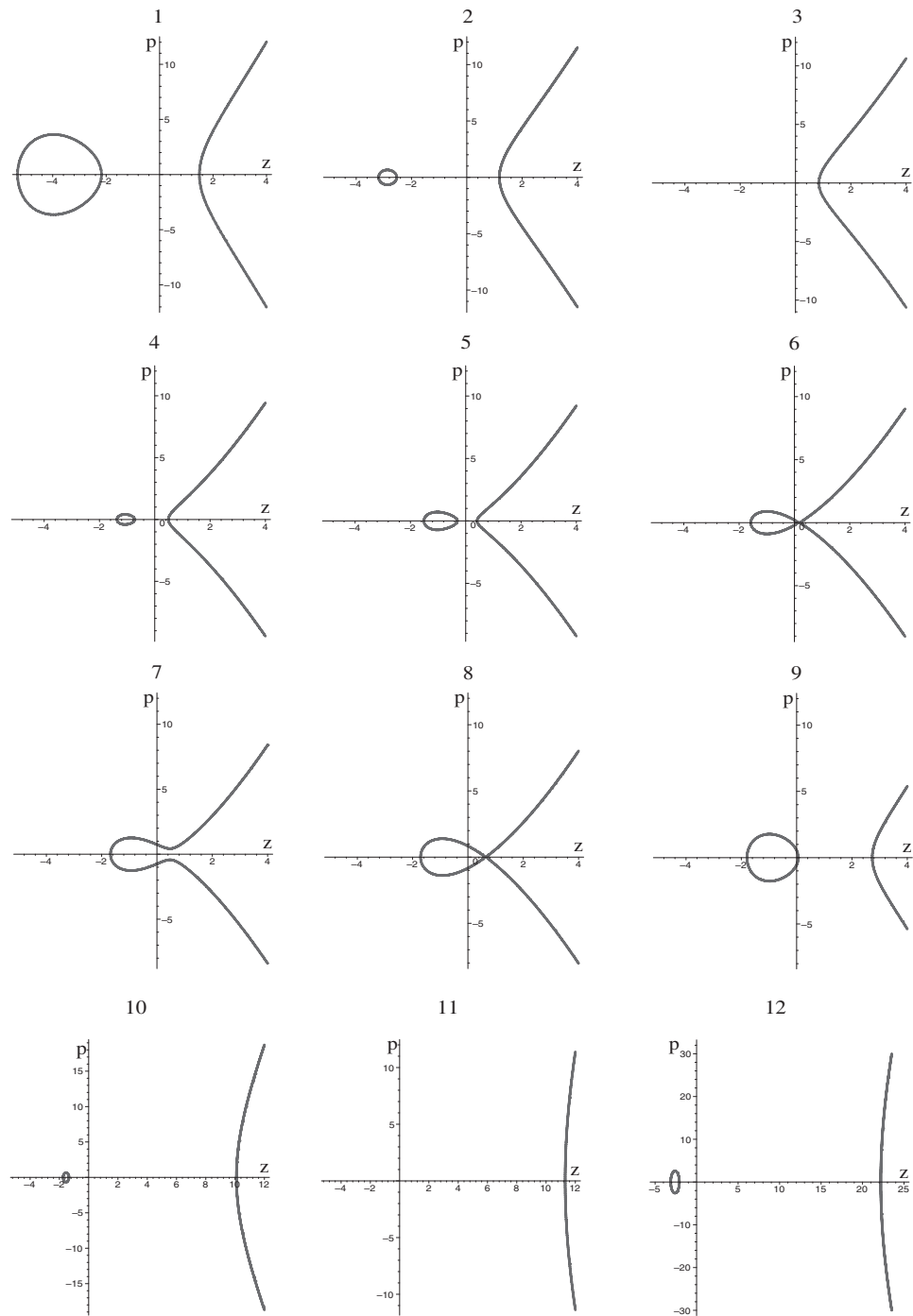


Figure 10. First plot: $x = 1.3, t = -2$; second plot: $x = 1.3, t = -1.27$; third plot: $x = 1.3, t = -0.5$; fourth plot: $x = 1.3, t = 0.2$; fifth plot: $x = 1.3, t = 0.3$; sixth plot: $x = 1.3, t = 0.3871$; seventh plot: $x = 1.3, t = 0.65$; eighth plot: $x = 1.3, t = 0.786$; ninth plot: $x = 1.3, t = 1.5$; tenth plot: $x = 1.3, t = 4.5$; eleventh plot: $x = 1.3, t = 5$; twelfth plot: $x = 1.3, t = 9.5$.

is degenerated if $4g_2^3 + 27g_3^2 = 0$. In this case, the curve (52) can be presented in the form

$$p^2 - (z + 2u)(z - u)^2 = 0, \tag{53}$$

where u is the uniformizing variable defined by $g_2 = -3u^2$ and $g_3 = 2u^3$.

The singular cubic curve (53) has a well-known rational parameterization

$$\begin{aligned} z &= q^2 - 2u, \\ p &= q^3 - 3uq. \end{aligned} \tag{54}$$

Indeed, introducing the variable q by relation

$$p = q(z - u), \tag{55}$$

one represents equation (53) as

$$q^2 - (z + 2u) = 0. \tag{56}$$

Two equations (55) and (56) are obviously equivalent to (54).

The parameterization (54) clearly indicates the interrelation between the degenerated cubic curve (53) and the Burgers–Hopf (dKdV) equation:

$$u_x = 3uu_y. \tag{57}$$

Equation (57) is known to be equivalent to the compatibility condition for two Hamilton–Jacobi equations (see e.g. [16, 23, 41])

$$\begin{aligned} (\partial_y S)^2 - 2u &= z, \\ \partial_x S &= (\partial_y S)^3 - 3u\partial_y S. \end{aligned} \tag{58}$$

In our approach, equation (57) describes the Hamiltonian deformations of the parabola $f = z - q^2 + 2u(y, x)$ generated by the Hamiltonian $H = q^3 - 3u(y, x)q$ or coisotropic deformations of the common points of the curves [21]

$$\begin{aligned} f &= z - q^2 + 2u(y, x) = 0, \\ g &= p + q^3 - 3uq = 0, \end{aligned} \tag{59}$$

with the standard Poisson bracket $\{f, g\} = \partial_y f \partial_q g - \partial_y g \partial_q f + \partial_x f \partial_p g - \partial_x g \partial_p f$. Thus, the auxiliary problem (59) for equation (57) represents nothing but the parameterization (54) of the cubic curve (53). So, any solution $u(x, t)$ of the Burgers–Hopf equation (57) provides us with the family of cubic curves (53) parameterized by the variables x and y . The solution

$$u = \frac{y_0 - y}{3x} \tag{60}$$

of equation (57) (y_0 is a constant) describes simple Hamiltonian deformation of the degenerated cubic (53) shown in figure 11. The deformation (60) connects singular points of different regimes shown in figures 4–6, i.e. curve with a separate point, cusp and node. This observation shows the relevance of the Burgers–Hopf (dKdV) equation for the description of the singular sector in the process of deformation of elliptic curves. Within the study of the Laplace growth process, this fact has been observed earlier in papers [5, 24, 28].

The inverse process of desingularization on the degenerated cubic curve, i.e. the passage from the curve (53) to the curve (33), can be viewed at least in two different ways. One is based on the Birkhoff stratification for the Burgers–Hopf hierarchy. In this approach, the passage from (53) to (33) is associated with the transition from the big cell of the Sato Grassmannian to the first stratum [17]. Another approach discussed in [5, 24, 28] suggests the dispersive regularization. Possible interconnection between these two approaches remains an open problem.

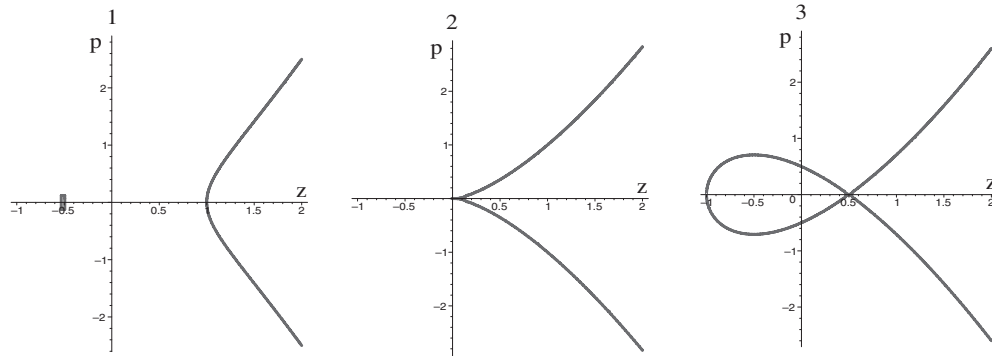


Figure 11. First plot: $y = 1.5 y_0, x = 1/3$ (u negative); second plot: $y = y_0, x = 1/3$ (u null); third plot: $y = 0.5 y_0, x = 1/3$ (u positive). In the first plot, there is also an isolated point in $(z, p) = (-0.5, 0)$.

6. Deformations of plane quintic

The analysis of the process of formation of singularities and bubbles presented in sections 2–5 can be extended to high-order (higher genus) plane algebraic curves. Such analysis becomes more involved but, on the other hand, it shows the presence of new phenomena. For instance, for the fifth-order curve given by

$$f = p^2 - \left(z^5 + \sum_{i=0}^4 u_i z^i \right) = 0 \tag{61}$$

in addition to the transition regime described above one has a novel regime of creation of two bubbles either one after another or one from another or simultaneously. The choice of the Hamiltonian similar to that used in the genus 1 case, i.e.

$$H = \left(\frac{u_4}{2} - z \right) p, \tag{62}$$

gives rise to the following system ($x_1 = x$) (see e.g. [17]):

$$\begin{aligned} \partial_t u_4 &= u_{3x} - \frac{2}{2} u_{4x} u_4, \\ \partial_t u_i &= u_{i-1x} - \frac{1}{2} u_{ix} u_4 - u_{4x} u_i, \quad i = 1, 2, 3, \\ \partial_t u_0 &= -\frac{1}{2} u_{0x} u_4 - u_{4x} u_0. \end{aligned} \tag{63}$$

In this case, equation (2) becomes

$$\partial_t f + \{f, H\} = -u_{4x} f. \tag{64}$$

The simplest polynomial solution of this system is

$$\begin{aligned} u_4 &= C_4, \\ u_3 &= C_3 + A_2 t, \\ u_2 &= C_2 + \left(A_1 - \frac{1}{2} A_2 C_4 \right) t + A_2 x, \\ u_1 &= C_1 + \left(A_0 - \frac{1}{2} A_1 C_4 \right) t + A_1 x, \\ u_0 &= C_0 - \frac{1}{2} A_0 C_4 t + A_0 x. \end{aligned} \tag{65}$$

Even this very simple example depicted in figure 12 in the particular case $A_0 = A_1 = A_2 = C_4 = 1, C_0 = -12, C_1 = C_2 = 0, C_3 = 3$ shows a new phenomenon, i.e. the formation

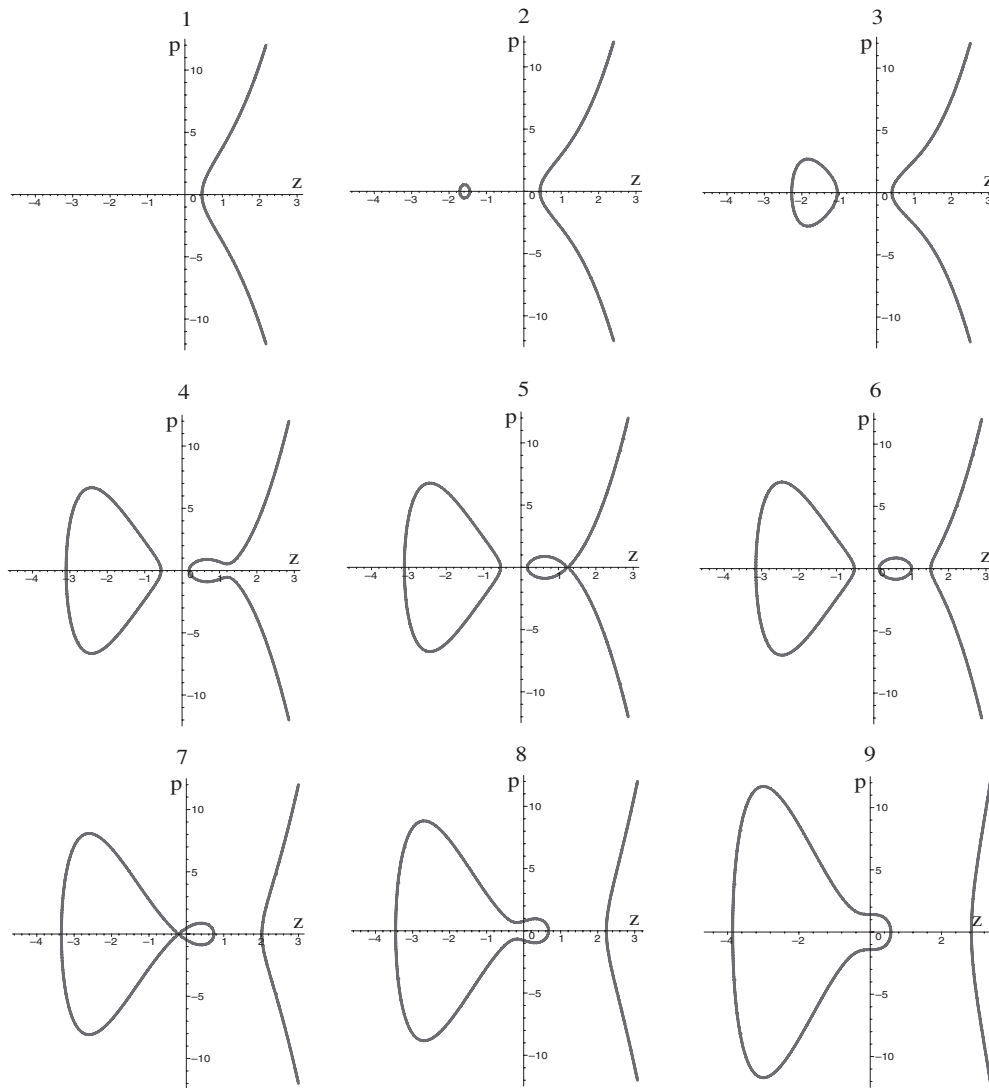


Figure 12. First plot: $x = 7, t = 0$; second plot: $x = 7, t = -3.6$; third plot: $x = 7, t = -5$; fourth plot: $x = 7, t = -9$; fifth plot: $x = 7, t = -9.11$; sixth plot: $x = 7, t = -9.3$; seventh plot: $x = 7, t = -10.45$; eighth plot: $x = 7, t = -11.2$; ninth plot: $x = 7, t = -14$.

of two bubbles. There are also various regimes of formation of singularities and bubbles (including the cusp (5, 2)) described by the system (63).

For higher order hyperelliptic curves of genus g , one observes processes of formation of g bubbles.

The Burgers–Hopf (dKdV) hierarchy, again, is relevant for the description of the singular sectors of these transition regimes. The process of regularization of the corresponding singular curves via the transition to higher Birkhoff strata in Sato Grassmannian and its connection with results of the papers [5, 24, 28] will be discussed elsewhere.

7. Perspectives

Hamiltonian deformations of plane algebraic curves introduced in this paper can be used for the study of a special class of deformations of the, so-called, quadrature (algebraic) domains on the plane. Quadrature domains (\mathcal{D}) on the plane are those for which an integral of any function ϕ over this domain is the linear superposition of values of ϕ and its derivatives in a finite number of points (see e.g. [1, 35]). Such domains show up in many problems of mathematics and fluid mechanics (see e.g. [12, 24, 30, 34, 35]). A particular property of quadrature domains is that their boundaries $\partial\mathcal{D}$ are algebraic curves. So any Hamiltonian deformation of the boundary $\partial\mathcal{D}$ generates a special deformation of quadrature domains. One can refer to such deformations as the Hamiltonian (or coisotropic) deformations of quadrature domains. The study of the properties of such deformations, for instance the analysis of deformations of the quadrature domain data, would be of interest.

We note that the Hamiltonian deformations can be defined for surfaces and hypersurfaces too. Indeed, let a hypersurface in \mathbb{C}^n be defined by the equation

$$f(p_1, \dots, p_n) = 0. \quad (66)$$

Introducing deformation parameters x_1, \dots, x_n, t , one may define Hamiltonian deformations of a hypersurface (66) by the formulae (2)–(4) passing from two to n variables p_j and x_j . For algebraic surfaces ($n = 3$) and hypersurfaces, such deformations are described by the systems of equations of hydrodynamical type. Properties of such systems and the corresponding Hamiltonian deformations are worth studying.

These problems and also the comparison of Hamiltonian deformations of plane curves and surfaces with those studied in the papers [5, 6, 14, 24–27, 30–32, 38–40] and with classical deformation theory (see e.g. [2, 11, 36]) will be addressed in future publications.

Acknowledgments

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